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BASES–COBASES GRAPHS AND POLYTOPES OF MATROIDS

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Let M be a *block matroid* (i.e. a matroid whose ground set E is the disjoint union of two bases). We associate with M two objects:
1. The *bases–cobases graph* $G=G(M,M^*)$ having as vertices the bases B of M for which the complement $E\setminus B$ is also a base, and as edges the unordered pairs (B,B') of such bases differing exactly by two elements.
2. The *polytope of the bases–cobases* $K=K(M,M^*)$ whose extreme points are the incidence vectors of the bases of M whose complement is also a base.

We prove that, if M is graphic (or cographic), the distance between any two vertices of G corresponding to disjoint bases is equal to the rank of M (generalizing a result of [10]).

Concerning the polytope we prove that K is an hypercube if and only if $\dim(K)=\text{rank}(M)$. A constructive characterization of the class of matroids realizing this equality is given.

1. Introduction and Notations

We shall assume familiarity with matroid theory (see [13], [14] and [15] whose notation and terminology we will follow with minor changes). If A is a subset of a set E and x is an element of E we will write $A+x$ and $A-x$ for $A\cup\{x\}$ and $A\setminus\{x\}$, respectively. If B is a base [resp. a cobase] of a matroid M and $x\notin B$ we note $C(B,x)$ [resp. $C^*(B,x)$] the (unique) circuit [resp. cocircuit] contained in $B\cup\{x\}$.

Let M be a block matroid of rank r. Concerning the base–cobases graph of M, $G(M,M^*)$, two conjectures have been formulated:

Conjecture 1.1. ([5]) *$G(M,M^*)$ is connected.*

Conjecture 1.2. ([9], [10]) *There is a bijection O from E onto $\{1,2,\dots,2r\}$ such that*

$$B(i)=\{O^{-1}(k)\ :\ k=i,i-1,\dots,i-r-1\ (\text{mod } 2r)\}$$

is a base of M for any i, $1\le i\le 2r$.

The last one can be reformulated as follows:

Conjecture 1.3. *$G(M,M^*)$ has two vertices corresponding to disjoint bases joined by a path of length r.*

We notice that the second conjecture has an interesting application to Feynman integrals. In fact, a matroid M on a ground set E for which the second conjecture

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holds is a *baseable* matroid, i.e. there are $2r$ bases of M such that each element e of E belongs to exactly r of them. And, as it has been proved in [8], this is a sufficient condition for the convergence of a generalization of the Feynman integral.

The authors of both conjectures proved them for graphic (and cographic) matroids and the first one was also proved for transversal matroids (see [4]). Theorem 1.6 below proves for graphic matroids the following conjecture (stronger than the second one):

Conjecture 1.4. *If M is a block matroid, any pair of vertices of $G(M,M^*)$ corresponding to disjoint bases is joined by a path of length r.*

Notice that it is not true for $G(M,M^*)$, as it was for the *basis graph*, that the distance in $G(M,M^*)$ between any two bases–cobases B and B' is equal to $r-|B\cap B'|$. Not even if M is a graphic matroid (see [4] for a counter-example in K_4).

Conjectures 1.1 and 1.4 suggest the question of making a good estimation of the diameter of $G(M,M^*)$.

Conjecture 1.5. ([7]) *If G is the bases–cobases graph of a block matroid of rank r then the diameter of G is equal to r.*

Theorem 1.6. *If M is a graphic block matroid any pair of vertices of $G(M,M^*)$ corresponding to complementary bases is joined by a path of length* $\operatorname{rank}(M)$.

Proof. We shall use induction on $\operatorname{rank}(M)$. If $\operatorname{rank}(M)=1$ the result is trivial; thus, suppose that it is true for any graphic block matroid whose *rank* is lower than r and let M be a graphic block matroid and $\operatorname{rank}(M)=r$.

As $M=M(G)$ is graphic, there is a vertex in G whose degree is two or three. Let C^* be the correspondent cocircuit in M.

If $C^*=\{a,b\}$ the set of base–cobases is the same for the matroids M and $M\setminus\{a,b\}\oplus M^*(\{a,b\})$ and the theorem is clearly a consequence of the induction hypothesis.

If $C^*=\{a,b,c\}$ let B and B' be two disjoint bases of M. We can assume, without loss of generality, that $B\cap C^*=\{a,b\}$ and that $a\in C(B,c)$. In this case $B-a$ and $B'-c$ are complementary base–cobases of the matroid $M/a\backslash c$ whose rank is $r-1$. From the induction hypothesis it follows that there are r base-cobases of this matroid, $B-a=B_0,B_1,\ldots,B_{r-1}=B'-c$, such that $|B_i\cap B_{i+1}|=r-2$ for any i, $0\le i\le r-2$.

We claim that, for each i, $0\le i\le r-1$, if $b\in B_i$ [resp. $b\notin B_i$], B_i+a [resp. B_i+c] is a base–cobase of M. Furthermore, if $b\in B_i$ then one of the sets B_i+c and $((B_i+a)-b)+c$ is also a base–cobase of M.

We begin by noticing that since B_i is a base [resp. a cobase] of $M/a\setminus c$ then B_i+a[resp. B_i+c] is a base [resp. a cobase] of M. Thus, when $b\in B_i$, $C^*(B_i+c,a)=\{a,b,c\}$ and B_i+a is also a cobase of M. Also, as $|C(B_i+a,c)\cap\{a,b\}|=1$, one of the sets B_i+c or $((B_i+a)-b)+c$ is a base of M, depending on which of the elements a or b, respectively, belongs to the circuit $C(B_i+a,c)$. Otherwise $b\notin B_i$ and so a is necessarily an element of $C(B_i+a,c)$, which makes also B_i+c a base of M.

Now we can define the path joining B and B' in $G(M)$ by the following sequence of vertices: $B=B'_0=B_0+a$, $B'_1=B_1+a,\ldots,B'_s=B_s+a$, B'_{s+1}, $B'_{s+2}=B_{s+1}+c$, $\ldots$, $B'_r=B_{r-1}+c=B'$ where s is the last integer such that $b\in B_s$ and B'_{s+1} is

B_s+c or $((B_s-b)+a)+c$, according to convenience. It is clear that, for each i, $0\leq i\leq r-1$, $|B'_i\cap B'_{i+1}|=r-1$ and the theorem follows. ∎

In the second part of this paper we initiate the study of the polytope $K = K(M,M^*)$ (see [1] and [6] for related (0,1)-polytopes).

2. The bases–cobases polytope

Let M be a matroid on a ground set E. To each base B of M we associate a vector v_B in $\mathbb{R}^E$; v_B is the incidence vector of B, i.e. $(v_B)_a=1$ if $a\in B$ and $(v_B)_a=0$ if $a\in E\setminus B$. If no confusion arises we will write B for v_B.

It is well known that these vectors are the extremal points of the bases polytope of M, $K(M)$, which is defined to be their convex hull ([2]). Besides, it is implicit in [12] that the 1-skeleton of this polytope determines it. We remark that, s being the number of connected components of $M=M_1\oplus\ldots\oplus M_s$, the bases polytope $K(M)$ is a translated from the product $K(M'_1)\times\ldots\times K(M'_s)$, where M'_i is for each i, $1\leq i\leq s$, the matroid M_i or its dual M_i^*.

If M is a block matroid it is clear that the bases–cobases of M are the extremal points of the polytope $K(M,M^*) = K(M)\cap K(M^*)$ whose study we begin by proving the next theorem. Notice that generally the graph $G(M,M^*)$ is not the 1-skeleton of the polytope $K(M,M^*)$ — take, for example, the cycle matroid M of the graph K_4, represented in Figure 1, and its bases–cobases $B_1=\{1,4,5\}$ and $B_2=\{3,4,6\}$;v_{B_1} and v_{B_2} are vertices of squared 2-faces of both $K(M)$ and $K(M^*)$ whose intersection is an edge of $K(M,M^*)$ that doesn't belong to $G(M,M^*)$.

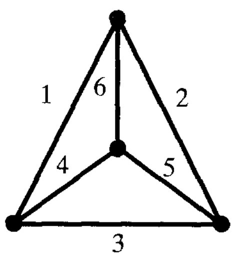

Fig. 1

Proposition 2.1. *Let M be a block matroid with* $\mathrm{rank}(M)=r$. *Then the dimension of the polytope $K(M,M^*)$ is greater or equal to r and the following conditions are equivalent:*
2.1.1 $\dim(K(M,M^*))=r$.
2.1.2 *$K(M,M^*)$ is an hypercube.*

Proof. Let B be any base–cobase of M. For each $x\in E\setminus B$ there is a $y(x)=y_x\in B$ such that $B-y_x+x$ is also a base–cobase of M (there exists $y_x\in C(B,x)\cap C^*(B,x)\setminus\{x\}$ because $|C\cap D|\neq 1$ for every circuit C and every cocircuit D). It is clear that the set $V=\{v_B\}\cup\{v_{B-y_x+x}:x\in E\setminus B\}$ of $r+1$ points in $\mathbb{R}^E$ is affinely independent and this proves our first statement.

2.1.1 $\Rightarrow$ 2.1.2

Note that, since 2.1.1 is true, the bases–cobases $B - y_x + x$ that we have just considered define, when x runs through $E \setminus B$, a partition of the ground set $E(M)$ in two-element subsets $\{x, y_x\}$: Suppose not; then there will be an element b in B not used in the construction of the r bases–cobases $B - y_x + x$. In this case, there is $a \in E \setminus B$ such that $B - b + a$ is a base–cobase and its incidence vector is affinely independent from the set V, contradicting condition 2.1.1 Now, let $\{x_1, y_1\} \uplus \{x_2, y_2\} \uplus \ldots \uplus \{x_r, y_r\} = E$ be the considered partition. We claim that, if B' is any base–cobase of M, $|B' \cap \{x_i, y_i\}| = 1$ for each i, $1 \leq i \leq r$ and thus $K(M, M^*)$ is the r-cube. In fact, if for some B' and some $i \in \{1, 2 \ldots, r\}$: $|B' \cap \{x_i, y_i\}| = 0$ then $v_{B'}$ would not be in the affine hull of the incidence vectors of the $r+1$ bases–cobases in V, contradicting the fact that they are generators of the polytope $K(M, M^*)$.

2.1.2 $\Rightarrow$ 2.1.1

Suppose now that $K = K(M, M^*)$ is a hypercube in $\mathbb{R}^E$. To each pair of parallel facets $\{F, F'\}$ of K we associate a pair of elements in E, $\{a, a'\}$, such that $B \in F$ [resp. $B \in F'$] if and only if $a \in B$ and $a' \notin B$ [resp. $a \notin B$ and $a' \in B$]. Observe that, being K a hypercube, all his edges are alike, i.e. they all are of the type $[B, B']$ where B and B' are bases–cobases and $|B \setminus B'| = 1$. So, if $B \in F$ then $B - a + a' \in F'$ and the correspondence between the facets of the polytope K and the elements of E is clearly a bijection, finishing our proof. ∎

This proof suggests the following definition:

Definition 2.2. A Block matroid is *weakly reducible* if there is a partition, $\{a_1, a_2\}, \ldots, \{a_{2r-1}, a_{2r}\}$, of the ground set $E(M)$ such that each base–cobase of M has nonempty intersection with any set of the partition.

Remark 2.3. Notice that for a weakly reducible block matroid M, any subset with two elements of $E(M)$ and nonempty intersection with every base–cobase is, necessarily, one of the sets of the partition. Besides, it follows from Proposition 2.1 that, if M is a block matroid, the following assertions are equivalent:
2.3.1 M is weakly reducible.
2.3.2 $\dim(K(M, M^*)) = r$.
2.3.3 $K(M, M^*)$ is a hypercube.

We remark that this definition is auto-dual, i.e., M is weakly reducible iff M^* is weakly reducible, and also that the partition of the ground set $E(M)$ of a weakly reducible matroid M is unique.

Definition 2.2 is useless if one wants to construct a matroid M satisfying 2.3.2 or 2.3.3, however it is possible to give a good characterization of these matroids.

Definition 2.4. A block matroid M is *strongly reducible* if there is an ordered partition, $\{a_1, a_2\}, \ldots, \{a_{2r-1}, a_{2r}\}$, of the ground set E satisfying the two following conditions:
2.4.1 The set $\{a_{2r-1}, a_{2r}\}$ is a circuit or a cocircuit of M.
2.4.2 For each k, $1 \leq k \leq r-1$, the set $\{a_{2k-1}, a_{2k}\}$ is a circuit or a cocircuit of the block matroid $M / \{a_{2k+2}, \ldots, a_{2r}\} \setminus \{a_{2k+1}, \ldots, a_{2r-1}\}$.

The reader can easily verify that a matroid strongly reducible is weakly reducible and that this new definition is also auto-dual. If a matroid is not weakly reducible we will say that it is *irreducible*. We also remark that for a block matroid M and a cocircuit [resp. a circuit] $\{a, b\}$ of M the pair of matroids (M, M^*) is reducible in the sense of [6] with respect to the ordered partition $E_1 = E(M) \setminus \{a,b\}$, $E_2 = \{a,b\}$ [resp. $E_1 = \{a,b\}, E_2 = E(M) \setminus \{a,b\}$].

Examples. 1. The cycle matroid of the graph obtained doubling all the edges of a tree is clearly strongly reducible.

2. The cycle matroid of the graph G in Figure 2 is strongly reducible with respect to the ordered partition: $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{7,8\}$. Notice that the graph G is not a series parallel graph.

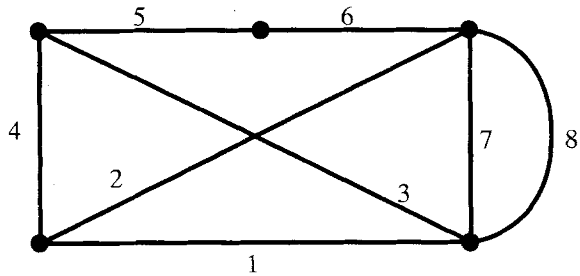

Fig. 2

The eight points matroid M represented in Figure 3 is strongly reducible with respect to the ordered partition: $\{1,2\}$, $\{3,4\}$, $\{5,6\}$, $\{7,8\}$ of its ground set. We remark that M is not a binary matroid.

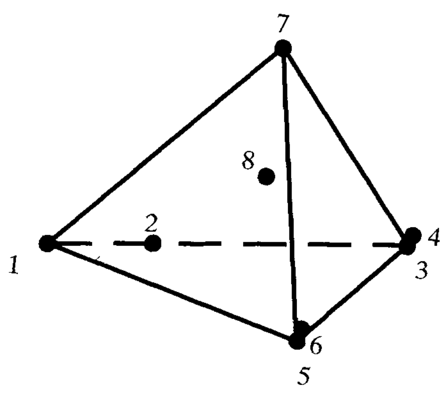

Fig. 3

Theorem 2.5. *Let M be a rank r block matroid; the following conditions are equivalent:*
2.5.1 *M is strongly reducible;*
2.5.2 $\dim(K(M,M^*)) = r$;
2.5.3 *$K(M,M^*)$ is an hypercube.*

Before proving this theorem we will show that Definition 2.4 is equivalent to the following:

Definition 2.6 A rank r block matroid M is strongly reducible if there is an ordered partition $\{a_1,a_2\},\dots,\{a_{2r-1},a_2r\}$ of its ground set satisfying:
2.6.1 $\{a_{2r-1},a_{2r}\}$ is a circuit of M;
2.6.2 for each k, $1 \leq k \leq r-1$, $\{a_{2k-1},a_{2k}\}$ is a circuit of the block matroid $M/\{a_{2k+1},\dots,a_{2r}\}$.

Proposition 2.7. *Definitions 2.4 and 2.6 are equivalent.*

Proof of Proposition 2.7. We begin by noting that if $\{a,b\}$ is a circuit [resp. a cocircuit] of a matroid M then $M/a\setminus b = M/\{a,b\}$ [resp. $M\setminus a/b = M\setminus\{a,b\}$]. Thus it is clear that a matroid satisfying Definition 2.6 satisfies also Definition 2.4. To prove the non-trivial implication 2.4 $\Rightarrow$ 2.6 we proceed by induction on $r = \mathrm{rank}(M)$. The implication is clear if $\mathrm{rank}(M) = 1$ and we will suppose it is true whenever $1 \leq \mathrm{rank}(M) < r$. Now, let M be a block matroid of rank r satisfying Definition 2.4. We can suppose that $\{a_{2r-1},a_{2r}\}$ is a cocircuit of M, otherwise the result follows from the induction hypothesis. In this case $\{a_{2r-1},a_{2r}\}$ is also a cocircuit of the matroid $M' = M/(E\setminus\{a_{2r-1},a_{2r}\})$ and thus a circuit of M' since $\mathrm{rank}(M') = 1$. On the other hand, using the induction hypothesis, one can reorder the partition sets $\{a_1,a_2\},\dots,\{a_{2r-3},a_{2r-2}\}$ is such a way that the block matroid $M'' = M\setminus\{a_{2r-1},a_{2r}\}$ satisfies Definition 2.6. Call this new order $\{a'_1,a'_2\},\dots,\{a'_{2r-3},a'_{2r-2}\}$. Then $\{a'_{2r-3},a'_{2r-2}\}$ is a circuit of M'' and so it is also a circuit of M. Besides, for each k, $1\leq k\leq r-2$, $\{a'_{2k-1},a'_{2k}\}$ is necessarily a circuit of $M/\{a'_{2k+1},\dots,a'_{2r-2}\}$. Therefore M satisfies conditions 2.6.1 and 2.6.2 for the ordered partition of its ground set: $\{a_{2r-1},a_{2r}\},\{a'_1,a'_2\},\dots,\{a'_{2r-3},a'_{2r-2}\}$. ∎

We remark that, since the notion of strongly reducible matroid is auto-dual, it follows from Proposition 2.7 that one can change in Definition 2.6 the word *circuit* for *cocircuit* and the operation *contraction* for *deletion* and necessarily obtain a third equivalent definition for strong reducibility which we will refer to as Definition 2.6*.

Proof of Theorem 2.5. By Proposition 2.1, it is sufficient to show that every weakly reducible matroid is also strongly reducible to prove the theorem. We will use induction on the rank of the matroid to prove it. The statement is trivial if $\mathrm{rank}(M) = 1,2$ and we will assume it is true when $2 < \mathrm{rank}(M) < r$. So, take a rank r matroid M which is weakly reducible and let $\{a_1,a_2\},\dots,\{a_{2r-1},a_{2r}\}$ be the partition of its ground set $E(M)$ that satisfies Definition 2.3. Notice that, since M is weakly reducible, for every partition-set $\{a_{2i-1},a_{2i}\}$, $1\leq i\leq r$, the matroid $M/a_{2i-1}\setminus a_{2i}$ is also weakly reducible. Thus, if there is one set in the partition of $E(M)$ which is either a circuit or a cocircuit our assertion follows from the induction hypothesis.

Suppose now, for a contradiction, that all the partition-sets are neither circuits nor cocircuits of M and consider the matroids $M' = M/a_{2r-1}\setminus a_{2r}$ and $M'' = M/a_{2r}\setminus a_{2r-1}$. They both have rank $r-1$ and so, by induction hypothesis, they both are strongly reducible. Using the fact that M' [resp. M''] satisfies Definition 2.6 [resp. Definition 2.6*] and the uniqueness of the partition, one can say that one of the partition-sets $\{x,y\}$ is a circuit of M' [resp. one of the partition-sets $\{z,w\}$ is a cocircuit of M'']. Then under our supposition, $A=\{x,y,a_{2r-1}\}$ must be a circuit of M and $B=\{z,w,a_{2r-1}\}$ must be a cocircuit of M. As $a_{2r-1}\in A\cap B$ then $|A\cap B|\geq 2$

and so A and B must be the same set because the partition $\{a_1,a_2\},\dots,\{a_{2r-1},a_{2r}\}$ of $E(M)$ is unique. So $B = A = \{x,y,a_{2r-1}\}$ is a circuit and a cocircuit of M. Now, take the matroid $M_1' = M/x \setminus y$ which again is strongly reducible and let $\{b_1,b_2\},\dots,\{b_{2r-3},b_{2r-2}\}$ be the ordered partition of $E \setminus \{x,y\}$ for which M_1' satisfies Definition 2.6. As $\text{rank}(M_1') \geq 2, \{b_1,b_2\}$ is different from $\{b_{2r-3},b_{2r-2}\}$ and thus $\{b_1,b_2,x\}$ and $\{b_{2r-3},b_{2r-2},x\}$ are two different circuits of M. Noticing that $\{b_1,b_2\}$ is a circuit of the rank 1 matroid $M_1''=M_1'/(E\setminus\{b_1,b_2,x,y\})$, $\{b_1,b_2\}$ is also a cocircuit of M_1'' and so a cocircuit of M_1'. By hypothesis none of the partition sets is neither a circuit nor a cocircuit and thus $\{b_1,b_2,y\}$ must be a cocircuit of M. As $x \in \{b_{2r-3},b_{2r-2},x\} \cap \{x,y,a_{2r-1}\}$ and $y \in \{x,y,a_{2r-1}\} \cap \{b_1,b_2,y\}$ we get the contradiction $\{b_{2r-3},b_{2r-2}\} = \{a_{2r-1},a_{2r}\}$ and $\{b_1,b_2\} = \{a_{2r-1},a_{2r}\}$. The theorem follows. ∎

We will complete this paper stating some propositions relative to a restrict class of matroids. We begin with a result that is still true in general:

Proposition 2.8. *If M is a strongly reducible block matroid of rank r then there is a r-face in $K(M)$ which is a hypercube.*

Proof. It is well known [3] that for any matroid M and any subset A of its ground set $E(M), K(M(A) \oplus M/A)$ is a face of $K(M)$. Now let M be strongly reducible and consider the matroid $N=N_1(a_1,a_2)\oplus\dots\oplus N_r(a_{2r-1},a_{2r})$, Where N_i, $1 \leq i \leq r$, is a rank 1 block matroid and $\{a_1,a_2\},\dots,\{a_{2r-1},a_{2r}\}$ is the ordered partition of $E(M)$ associated to the strong reducibility of M. It is clear that $K(N)$ is a r-cube and it follows from the iterative use of our first assertion and from Definition 2.6 that $K(N)$ is a face of $K(M)$. ∎

Two questions can be made concerning this last proposition:

(i) In a given rank r matroid can there exist two r-faces of $K(M)$ that are r-cubes?

(ii) Is the converse of Proposition 2.8 true?

Next theorem answers to these two questions whenever M is a binary matroid. Before stating it we recall a definition: let M, N be two matroids with the same rank and ground set; we say that there is a *rank preserving map* between M and N (and we denote it $M \xrightarrow{rp} N$) if all the bases of N are bases of M (see [11] and [14], Chap.9).

Theorem 2.9. *Let M be a binary block matroid of rank r; then the following statements are equivalent:*

2.9.1 *M is strongly reducible.*

2.9.2 *There is a r-face in $K(M)$ which is a hypercube.*

2.9.3 *There is a matroid N which is the direct sum of r pairs of parallel elements and such that $M \xrightarrow{rp} N$.*

Proof. 2.9.1 ⇒ 2.9.2 has already been proved and 2.9.2 ⇒ 2.9.3 is trivial, so it is enough to prove 2.9.3 ⇒ 2.9.1. Being M a binary matroid that satisfies 2.9.3 it follows from a Lucas' result ([14], Lemma 9.4.6) that for a certain pair $\{a,b\}$ of parallel elements $M(\{a,b\}) = N(\{a,b\})$. Then it is a clear that $M/\{a,b\} \xrightarrow{rp} N/\{a,b\}$ and the proof can be easily completed using induction. ∎

The following is an immediate consequence of last theorem:

Corollary 2.10. *Let M be a binary block matroid of rank r; if there is a r-face in $K(M)$ which is a hypercube then it is unique.*

We finish with a result for graphic matroids:

Propostion 2.11. *Let $M=M(G)$ be a graphic block matroid; the number of vertices of $K(M,M^*)$ is greater or equal to 2^r with equality if and only if $K(M,M^*)$ is a hypercube.*

Proof. We prove the first statement by induction on the rank(M). If rank$(M)=1$, G must be a two edges cycle and there is nothing to prove. Suppose now that this statement is true if rank$(M)<r$ and let M be a rank r graphic block matroid. Let $\{a,b\}$ be a cocircuit of $M(G)$, i.e., a and b are the edges incident with a certain vertex of G. Then $K(M,M^*)=K(N,N^*)$ where $N=M\setminus\{a,b\}\oplus M^*(\{a,b\})$ and the statement follows from the induction hypothesis.

Suppose now that G has no vertices of degree 2; thus there is a cocircuit $\{a,b,c\}$ with three elements in $M(G)$. Take any base–cobase B of M and suppose, without loosing generality, that $B\cap\{a,b,c\}=\{a,b\}$ and that $b\in C(B,c)$ (i.e. $B-b+c$ is a base–cobase of M). Then $M'=M/b\setminus c$ and $M''=M\setminus b/c$ are block matroids. Let B_0 be any base–cobase of M' [resp. M'']. If $a\in B_0$ then B_0+b [resp. B_0+c] is a base–cobase of M. If $a\notin B_0$ then B_0+c [resp. B_0+b] is a base–cobase of M. Using this argument and the induction hypothesis we establish the first statement.

Notice that the same argument shows that, when M has exactly 2^r bases–cobases, all the bases–cobases of M have nonempty intersection with $\{b,c\}$. I.e., $K(M,M^*)=K(N_1,N_1^*)$ where N_1 is the direct sum of $M/b\setminus c$ and the (graphic) block matroid on $\{b,c\}$. From this assertion it is easy to prove by induction that if $K(M,M^*)$ has 2^r vertices if $K(M,M^*)$ is a r-cube. Now, using Theorem 2.1 the proof follows easily. ∎

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